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Multiple orthogonal polynomials

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Abstract

Results on multiple orthogonal polynomials will be surveyed. Multiple orthogonal polynomials are intimately related to Hermite–Padé approximants and often they are also called Hermite–Padé polynomials. Special attention will be paid to an application of multiple orthogonal polynomials and to analytic theory of two model families of general multiple orthogonal polynomials, referred to as Angelesco and Nikishin systems. Among the applications the number theory, special functions and spectral analysis of nonsymmetric band operators will be highlighted. In the analytic theory results and methods for the study of multiple orthogonal polynomials asymptotics will be reviewed. New results on strong asymptotics of multiple orthogonal polynomials for Nikishin system will be presented.

Keywords: Multiple orthogonality; Hermite–Padé approximants; asymptotics of orthogonal polynomials

1. Introduction

During the past 15 years there was a stable interest in the study of multiple orthogonal polynomials. There are several survey papers about the topic (see [16, 22, 46, 57]). Here we review the progress of the subject for the past five years, particularly in its application to various fields of mathematics and in the study of their analytic, asymptotic properties.

In the introduction below we present the basic definitions, show a connection with Hermite–Padé approximants, introduce examples of general multiple orthogonal polynomials such as Angelesco and Nikishin systems and consider polynomials orthogonal with respect to a varying weight, which play the key role in the study of the analytic properties of general multiple orthogonal polynomials.

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1.1. Sequences of multiple orthogonal polynomials

Definition 1.1. A polynomial $Q_{\boldsymbol{n}}(x)$ is called *multiple orthogonal polynomial* of a vector index

$$\boldsymbol{n}=(n_1,\ldots,n_p)\in\mathbb{N}_p \tag{1.1}$$

with respect to a vector of positive Borel measures, supported on the real axis

$$\boldsymbol{\mu}=(\mu_1,\ldots,\mu_p),\quad \operatorname{supp}\mu_\alpha\in\mathbb{R},\quad \alpha=1,\ldots,p \tag{1.2}$$

if it satisfies the following conditions:

$$(*)\quad \deg Q_{\boldsymbol{n}}\leqslant|\boldsymbol{n}|:=\sum_{\alpha=1}^{p}n_\alpha, \tag{1.3}$$

$Q_{\boldsymbol{n}}$:

$$(**)\quad \int Q_{\boldsymbol{n}}(x)x^{\nu}\,\mathrm{d}\mu_\alpha(x)=0,\quad \nu=0,\ldots,n_\alpha-1,\ \alpha=1,\ldots,p.$$

When $p=1$ the multiple orthogonal polynomial becomes the standard orthogonal polynomial $Q_n(x)$:

$$\deg Q_n=n\quad\text{and}\quad \int Q_n(x)x^{\nu}\,\mathrm{d}\mu(x)=0,\quad \nu=0,1,\ldots,n-1. \tag{1.3$'$}$$

The notion of multiple orthogonal polynomial can be generalized if we consider the non-Hermitian complex orthogonality with respect to a complex valued vector function

$$\boldsymbol{f}(z)=(f_1(z),\ldots,f_p(z))=\left(\sum_{j=0}^{\infty}\frac{f_{1,j}}{z^{j+1}},\ldots,\sum_{j=0}^{\infty}\frac{f_{p,j}}{z^{j+1}}\right) \tag{1.4}$$

on some contours Γ_α of the complex plane $\mathbb{C}$.

Thus Definition 1.1 becomes

Definition 1.1$'$. A polynomial $Q_{\boldsymbol{n}}$ is called multiple orthogonal polynomial of index (1.1) with respect to a complex weight (1.4) if it satisfies $(*)$ in (1.3) and

$$\oint_{\Gamma_\alpha} Q_{\boldsymbol{n}}(z)z^{\nu}f_\alpha(z)\,\mathrm{d}(z)=0,\quad \nu=0,\ldots,n_\alpha-1,\ \alpha=1,\ldots,p. \tag{1.5}$$

Usually multiple orthogonal polynomials are considered with the following sequences of indexes

(a) diagonal:

$$\boldsymbol{n}\in I_D \Leftrightarrow \boldsymbol{n}=(n,\ldots,n),\quad n\in\mathbb{N},\ |\boldsymbol{n}|=p\cdot n. \tag{1.6}$$

(b) Step-line:

$$\boldsymbol{n}\in I_{SL} \Leftrightarrow \boldsymbol{n}=(m+1,\ldots,m+1,m,\ldots,m),\quad m\in\mathbb{N},\ |\boldsymbol{n}|:=n\in\mathbb{N}. \tag{1.7}$$

Multiple orthogonal polynomials along the step-line are also called vector orthogonal polynomials (see [60]).

1.2. *Hermite–Padé (simultaneous) rational approximants*

From Definition 1.1′ it follows that if the components of the vector function (1.4) are analytic functions in a neighbourhood of infinity containing contours Γ_α, then there exists a set of polynomials $P_{\boldsymbol{n}}^{(\alpha)}$, $\alpha = 1, \ldots, p$ such that

$$\deg P_{\boldsymbol{n}}^{(\alpha)} \leqslant |\boldsymbol{n}| - 1, \quad Q_{\boldsymbol{n}}(z) f_\alpha(z) - P_{\boldsymbol{n}}^{(\alpha)}(z) = \mathrm{O}\left(\frac{1}{z^{n_\alpha+1}}\right) \tag{1.8}$$

and therefore

$$f_\alpha(z) - \frac{P_{\boldsymbol{n}}^{(\alpha)}(z)}{Q_{\boldsymbol{n}}(z)} = \mathrm{O}\left(\frac{1}{z^{|\boldsymbol{n}|+n_\alpha+1}}\right), \quad \alpha = 1, \ldots, p. \tag{1.8′}$$

Definition 1.2. A set of rational function

$$\pi = \left(\frac{P_{\boldsymbol{n}}^{(1)}}{Q_{\boldsymbol{n}}}, \ldots, \frac{P_{\boldsymbol{n}}^{(p)}}{Q_{\boldsymbol{n}}}\right) \tag{1.9}$$

satisfying $(*)$ in (1.3) and (1.8) is called *Hermite–Padé approximants of type* II or *simultaneous rational approximants* of index $\boldsymbol{n}$, (1.1) for the vector-function (1.4).

Thus, multiple orthogonal polynomials are also called Hermite–Padé Polynomials or simultaneous orthogonal polynomials.

If we consider the simultaneous rational approximants for the vector-function in which the components are the so-called *Markov functions*

$$\boldsymbol{f}(z) = \left(\int \frac{\mathrm{d}\mu_1(x)}{z - x}, \ldots, \int \frac{\mathrm{d}\mu_p(x)}{z - x}\right) \tag{1.10}$$

then the common denominator of approximants (1.9) becomes the polynomial introduced by Definition 1.1.

1.3. *Examples of general multiple orthogonal polynomials: Uniqueness*

In the theory of orthogonal polynomials (see [59]) there are two subjects: classical orthogonal polynomials and general orthogonal polynomials. Properties of classical orthogonal polynomials depend very strongly on the special weights of orthogonality and many of the properties disappear under a small perturbation of the weight. In contrast, properties of general orthogonal polynomials (which include the classical ones) remain valid. Similarly, for the multiple orthogonal polynomial the study of the special weights of the polynomials is separated from the results of their general character. We will consider multiple orthogonal polynomials for the special weights in the next section. Here we introduce two families of general multiple orthogonal polynomials.

First, we address the existence and uniqueness of the multiple orthogonal polynomials (1.3), (1.5).

Existence. The polynomial $Q_{\boldsymbol{n}}$ determined by Definition 1.1 always exists, because for its $|\boldsymbol{n}| + 1$ unknown coefficients orthogonality relations (1.3) give a system of $|\boldsymbol{n}|$ linear algebraic homogeneous equations, which always has a nontrivial solution.

Uniqueness. Unlike the usual orthogonal polynomial ($p=1$) defined by (1.3′)
(a) Q_n is determined uniquely up to a constant factor;
(b) $\deg Q_n = n$; (1.11)
(c) Q_n has n simple zeros which are all contained in the convex hull of supp.
for the multiple orthogonal polynomials $p>1$ these assertions are in general not true. The simplest example is when the vector of measures (1.2) has identical components.

However, there exist two general systems of measures (1.2) for which multiple orthogonal polynomials inherit these basic properties (1.11) of orthogonal polynomials. They are called Angelesco and Nikishin systems.

1.3.1. Angelesco system

Definition 1.3. A system of arbitrary measures (1.2) which satisfy the following conditions

$$\boldsymbol{\mu}: \begin{cases} \operatorname{supp}\mu_\alpha = F_\alpha = [a_\alpha, b_\alpha] \subset \mathbb{R}, \\ F_\alpha \cap F_\beta = \emptyset, \quad \alpha \neq \beta, \quad \alpha, \beta = 1, \ldots, p, \end{cases} \tag{1.12}$$

is called *Angelesco system.*

The multiple orthogonal polynomials with respect to the system (1.2), (3.2) have been introduced in 1918–23 by Angelesco [3–5] (in 1979 they have been rediscovered by Nikishin [42]). An immediate consequence of orthogonality relations (1.3) and conditions (1.12) is the following

Theorem 1.4 (Angelesco [4]). *The multiple orthogonal polynomial $Q_{\mathbf{n}}$ with respect to the Angelesco system has in the interior of each F_α exactly n_α simple zeros, $\alpha = 1, \ldots, p$.*

Corollary 1.5. *The multiple orthogonal polynomial $Q_{\mathbf{n}}$ with respect to the Angelesco system is determined by Definition* 1.1 *in the unique way, up to a constant factor.*

1.3.2. Nikishin system

Another general system of measures which guaranties the uniqueness for the multiple orthogonal polynomials has been introduced by Nikishin in [43]. This system is generated by a general set of positive Borel measures

$$\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_p) \tag{1.13}$$

satisfying the condition

$$\operatorname{supp}\sigma_\alpha = F_\alpha = [a_\alpha, b_\alpha] \subset \mathbb{R}, \quad F_\alpha \cap F_{\alpha+1} = \emptyset, \quad \alpha = 1, \ldots, p-1, \tag{1.14}$$

by the following procedure:

$$\begin{aligned} \mathrm{d}\mu_1(x) &= \mathrm{d}\sigma_1(x), \\ \mathrm{d}\mu_2(x) &= \left(\int_{F_2} \frac{\mathrm{d}\sigma_2(x_2)}{x - x_2} \right) \mathrm{d}\sigma_1(x), \\ \mathrm{d}\mu_3(x) &= \left(\int_{F_2} \int_{F_3} \left(\frac{\mathrm{d}\sigma_3(x_3)}{x_2 - x_3} \right) \frac{\mathrm{d}\sigma_2(x_2)}{x - x_2} \right) \mathrm{d}\sigma_1(x), \ldots \end{aligned} \tag{1.15}$$

If we use a notation from [29]

$$d\langle\sigma_1,\sigma_2\rangle := \int \frac{d\sigma_2(t)}{x-t}\, d\sigma_1(x), \quad d\langle\sigma_1,\sigma_2,\ldots,\sigma_k\rangle := d\langle\sigma_1,\langle\sigma_2,\ldots,\sigma_k\rangle\rangle,$$

then the definition of the Nikishin system becomes more compact.

Definition 1.6. A system of measures (1.2) which is generated by (1.13), (1.14) as

$$\boldsymbol{\mu}:\ \mu_1=\sigma_1,\ \mu_\alpha=\langle\sigma_1,\ldots,\sigma_\alpha\rangle, \quad \alpha=2,3,\ldots,p, \tag{1.15'}$$

is called a *Nikishin system.*

We note that contrary to the Markov system, all measures of the Nikishin system are supported by the same interval F_1.

Theorem 1.7 (Nikishin [43]; Driver and Stahl [23]). *If multiindex* (1.1) *satisfies*

$$n_\alpha \leqslant n_{\alpha-1}+1, \quad \alpha=2,3,\ldots,p, \tag{1.16}$$

then the multiple orthogonal polynomial $Q_{\boldsymbol{n}}$ with respect to the Nikishin system has in the interior of F_1 exactly $|\boldsymbol{n}|$ simple zeros. Therefore, under condition (1.16) *$Q_{\boldsymbol{n}}$ is defined by* (1.3) *in the unique way.*

Remark. Nikishin proved this theorem for the special case $\boldsymbol{n}\in I_{SL}$. For the indexes satisfying (1.16) the theorem has been proved by Driver and Stahl. The uniqueness of multiple orthogonal polynomials for Nikishin system with indexes not satisfying (1.16) is still an *open problem.*

1.4. Polynomials orthogonal with varying weight related to the multiple orthogonal polynomials

A system of polynomials $\boldsymbol{q_n}=\{q_{\alpha,\boldsymbol{n}}\}_1^p$ satisfying usual orthogonality relations but with respect to varying weight depending on the polynomials themselves, plays the key role for proving results about the uniqueness of multiple orthogonal and their asymptotic behaviour.

$$\boldsymbol{q_n}:\ \int_{F_\alpha} q_{\alpha,\boldsymbol{n}}(x)x^\nu w_\alpha(\boldsymbol{q_n};x)\,d\sigma_\alpha(x)=0, \quad \nu=0,\ldots,\deg q_\alpha-1,\ \alpha=1,\ldots,p. \tag{1.17}$$

For the Angelesco system, the set of polynomials $q_{\alpha,\boldsymbol{n}}$, $\alpha=1,\ldots,p$ is formed by the zeros of $Q_{\boldsymbol{n}}$ located at each interval F_α, $\alpha=1,\ldots,p$.

$$Q_{\boldsymbol{n}}(x)=\prod_{\alpha=1}^p q_{\alpha,\boldsymbol{n}}(x), \quad \deg q_{\alpha,\boldsymbol{n}}=n_\alpha. \tag{1.18}$$

These polynomials satisfy the usual orthogonality relations (1.17) with respect to the constant-signed varying (unknown) weight

$$w_\alpha(\boldsymbol{q_n};x)\,d\sigma_\alpha(x):=\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^p q_{\beta,\boldsymbol{n}}(x)\,d\mu_\alpha(x). \tag{1.19}$$

For the Nikishin system, such a set of polynomials $q_{\alpha,\mathbf{n}}$, $\alpha = 1, \dots, p$ is formed in much more sophisticated fashion. Indeed, it took several attempts for polishing this notion in [20, 23, 24, 44] before this construction was crystallized in [29] even for more general systems called mixed Angelesco–Nikishin systems.

Theorem 1.8 (see, for example, [29]). *Let $\mathbf{n}$ be a multi index* (1.1) *satisfying* (1.16) *and* σ_α, $\alpha = 1, \dots, p$ *be a set of positive Borel measures satisfying* (1.14). *Then*

(1) *There exists a system of polynomials*

$$\mathbf{q} := (q_1(x), \dots, q_p(x)), \quad \deg q_\alpha = \sum_{\beta=\alpha}^{p} n_\beta \tag{1.20}$$

which are defined in a unique way up to a constant factor, by the orthogonality relations

$$\int_{F_\alpha} q_\alpha(x) x^\nu w_\alpha(\mathbf{q}; x)\, \mathrm{d}\sigma_\alpha(x) = 0, \quad \nu = 0, 1, \dots, \deg q_\alpha - 1, \tag{1.21}$$

where the constant signed varying weight is

$$w_\alpha(\mathbf{q}; x) := \frac{h_\alpha(x)}{q_{\alpha-1}(x) q_{\alpha+1}(x)}, \quad q_0 \equiv q_{p+1} \equiv 1 \tag{1.22}$$

and function h_α is defined inductively

$$h_1(x) = 1, \quad h_\alpha(x) := \int_{F_{\alpha-1}} \frac{q_{\alpha-1}^2(t)}{x - t} \frac{h_{\alpha-1}(t)\, \mathrm{d}\sigma_{\alpha-1}(t)}{q_{\alpha-2}(t) q_\alpha(t)}, \quad \alpha = 2, \dots, p. \tag{1.23}$$

(2) *If* σ_α, $\alpha = 1, \dots, p$, *is a generating set of measures for Nikishin system* (1.15), *then multiple orthogonal polynomial* $Q_{\mathbf{n}}(x)$ *is*

$$Q_{\mathbf{n}}(x) = q_1(x). \tag{1.24}$$

Remark. Theorem 1.7 is an immediate corollary of Theorem 1.8.

2. Application of multiple orthogonal polynomials

In this section we review several application of multiple orthogonal polynomials. Apart from the traditional applications of Hermite–Padé polynomials to the constructive approximation of vector analytic functions (quadrature formulas, convergence of approximants) which we will consider in the next section, here we make a sketch about some new results in number theory, special functions and spectral theory of operators.

2.1. Number theory

Historically the first field for application of multiple orthogonal polynomials was number theory. Indeed, they were invented by C. Hermite (Hermite–Padé approximants) in 1873 especially for proving transcendency of "e" (see [30]). Since that time the Hermite method is the main tool for the investigation of arithmetic properties of real numbers. We just mention in this context the solution of the famous ancient problem about quadrature of the circle (i.e. transcendency of π) found by Lindemann in 1882. For more details about development of Hermite method see [45, 50].

Here we touch a problem about the arithmetic nature of the values of Riemann zeta-function

$$\zeta(s)=\sum_{v=1}^{\infty} v^{-s}$$

at the odd points $\zeta(2k+1)$. For the even points there exists the famous formula

$$\zeta(2k)=(-1)^{k-1}\frac{(2\pi)^{2k}}{2(2k!)}B_{2k} \tag{2.1}$$

(where B_{2k} is Bernoulli number), but for the odd points the first result has been obtained just in 1978.

Theorem 2.1 (Apery [6]). *$\zeta(3)$ is irrational!*

To prove this theorem Apery just presented a sequence of numbers

$$\begin{aligned} q_n &= \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}^2, \\ p_n &= \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}^2 \left[\sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3} \binom{n}{m}^{-1} \binom{n+m}{m}^{-1} \right] \end{aligned} \tag{2.2}$$

which after some normalisation give a sequence of integers $\tilde{q}_n$, $\tilde{p}_n$, such that

$$0 \neq r_n = \tilde{q}_n \zeta(3) - \tilde{p}_n \to 0, \quad n \to \infty.$$

It proves the theorem.

However, for some period of time since Apery declared his result it was a mystery where the numbers (2.2) came from. An explanation was given by Beukers and later by various constructions of Sorokin (see [19, 52]). It appears that Hermite–Padé approximants of Nikishin systems generate the Apery numbers (2.2). In order to highlight an importance of Nikishin systems (despite of their sophisticated definition) we present here one construction of Sorokin [52] for generating Apery numbers.

The starting point is Euler formula

$$\zeta(3)=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=n}^{\infty}\frac{1}{k^2}.$$

If we add to the Markov function

$$\hat{\mu}_3(z)=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=n}^{\infty}\frac{1}{k^2}\frac{1}{z^n},$$

other two Markov functions $\hat{\mu}_2(z)$, $\hat{\mu}_1(z)$, which are all together generated by some Nikishin system of measures

$$\hat{\mu}_3(z) = \int_0^1 \frac{\varphi(t)}{z-t}\,dt, \quad \hat{\mu}_2(z) = \int_0^1 \frac{\varepsilon(t)}{x-t}\,dt, \quad \hat{\mu}_1(z) = \int_0^1 \frac{dt}{x-t}$$

and

$$\frac{\varphi(x)}{1-x} = \int_{-\infty}^0 \frac{-e(t))}{x-t}\frac{dt}{1-t}, \quad \frac{\varepsilon(x)}{1-x} = \int_{-\infty}^0 \frac{1}{x-t}\frac{dt}{1-t}, \quad e(t) = \int_0^1 \frac{dt}{x-t}$$

then Hermite–Padé approximants for the system $\hat{\mu}_1(z)$, $\hat{\mu}_2(z)$, $\hat{\mu}_3(z)$ at $z=1$ after some normalisation produce Apery numbers (2.2).

It is still a challenging *open problem* to prove the irrationality of $\zeta(5)$ or the transcendency of $\zeta(3)$.

Developing the Hermite method for Nikishin systems of Markov functions, Sorokin proved

Theorem 2.2 (Sorokin [55]).

$$\left|\sum_{\nu=0}^{\rho} a_\nu \zeta(2)^\nu\right| > \frac{c(\rho)}{[\max_{\nu=0,\dots,\rho} |a_\nu|]^{\varkappa(\rho)}}$$

where $c(\rho)$ is effectively computable constant and $\varkappa(\rho) = \rho \cdot 45^\rho$.

From this result it follows not only that $\zeta(2)$ is transcendental (which is not a discovery because of (2.1)), but also some new estimation for the measure of transcendency of π.

2.2. Special functions theory

Since the beginning of the century (see for example [7]) plenty of examples of multiple orthogonal polynomials (1.3) and (1.5) with respect to the special weights have been studied. A nice collection of them are presented in the surveys and monography [21, 42, 46, 53]. However, until last years there was a lack of classification. For instance, in the previous survey on Hermite–Padé approximants in 1994, Stahl mentioned: "a comprehensive system of classification (of classical multiple orthogonal polynomials) is still missing". Here we report about some progress in this direction.

2.2.1. Moments generating functionals of class S

Firstly, we recall some basic facts from the classification of the classical and semiclassical orthogonal polynomials.

Let Φ, Ψ be polynomials

$$\Phi(z) = \prod (z - a_\nu), \quad \deg \Phi \geqslant 0, \quad \deg \Psi \geqslant 1, \; a_0 = 0, \tag{2.3}$$

let function $w(z)$ be a solution of the *Pearson differential equation*

$$w(z): \; (\Phi w)' + \Psi w = 0 \tag{2.4}$$

and contour Γ a Jordan arc or curve which satisfies

$$\Gamma:\ \Delta_\Gamma \Phi w P = 0, \quad \forall P \in \mathscr{P}, \tag{2.5}$$

for any polynomial P. (It can be seen that Γ starts and ends on the set of points $\{a_\nu\} \cup \{\infty\}$.)

Definition 2.3. A pair (w, Γ) which satisfies (2.3)–(2.5) is called semiclassical moments generating functional of class S, where

$$S = \max\{\deg \Phi - 2, \deg \Psi - 1\}. \tag{2.6}$$

The functional (w, Γ) generates moments by

$$w_\nu = \int_\Gamma z^\nu w(z)\,dz \tag{2.7}$$

Definition 2.4. A set $\{Q_n\}$ of polynomials of complex (non-Hermitian) orthogonality with respect to the functional (w, Γ) (see (2.3)–(2.5)) is called semiclassical orthogonal polynomials of class $S \geqslant 1$ if

$$Q_n\!:\ \deg Q_n \leqslant n, \int_\Gamma Q_n(z) z^\nu w(z)\,dz = 0, \quad \nu = 0, \ldots, n-1. \tag{2.8}$$

When $S = 0$, then the position of singularities of the differential equations (2.4) leads to a classification of different types of classical weights, which are

$$
\begin{array}{lll}
(J) & \Phi(z) = z(z-a), & w(z) = z^{\alpha_0}(z-a)^{\alpha_1}, \\
(L) & \Phi(z) = z, & w(z) = z^{\alpha} e^{\beta z}, \\
(H) & \Phi(z) = \text{const}, & w(z) = e^{\beta z^2}, \\
(B) & \Phi(z) = z^2, & w(z) = z^{\alpha} e^{\gamma/z}
\end{array}
$$

called Jacobi, Laguerre, Hermite, and Bessel weight functions, respectively. The correspondent paths of integration for the classical weights are Γ:

(J) Jordan arc joining the points 0 and a.

(L) Jordan arc joining the points 0 and ∞, such that in a neighborhood of infinity $\Re(\beta z) < 0$.

(H) Jordan curve in $\overline{\mathbb{C}}$, containing the point ∞, such that in a neighbourhood of infinity the two ends of the curve belongs to the different quarters of the plane where $\Re(\beta z^2) < 0$.

(B) Jordan curve in $\mathbb{C}$, containing the point 0, such that in neighbourhood of zero $\Re(\gamma/z) < 0$.

Classical orthogonal polynomials (with respect to the functional (w, Γ) of class $S = 0$) possess many remarkable properties like the existence of Rodrigues type formulae and a simple, exact expression (in terms of polynomials $\Phi(z)$ and $\Psi(z)$ only) for the coefficients of the differential and recurrence equations.

Semiclassical orthogonal polynomials of class $S \geqslant 1$ present more sophisticated set of special functions (see [36, 38, 41, 46]). They also satisfy some differential equation, but coefficients of these equations depend on parameters which are very difficult to determine. The situation is the same concerning the coefficients of the recurrence relations, they can be described as a solution of some nonlinear problem. Moreover, it is very difficult to classify them because when $S \geqslant 1$, for the semiclassical weight (2.9) there exist $S+1$ homotopically different classes or Γ, satisfying (2.5), (see [18, 39, 40]) and the properties of the polynomials orthogonal with respect to the same semiclassical weight $w(z)$ but for different Γ sometimes are completely different.

2.2.2. Semiclassical multiple orthogonal polynomials of class S: Classification

The last remark from the previous subsection gives an idea to consider multiple orthogonality on the contours Γ_α, $\alpha = 1, \dots, s+1$, instead of the usual orthogonality relation for the semiclassical weights $w(z)$ for $S \geqslant 1$. In [14] a notion of semiclassical multiple orthogonal polynomials of class S is introduced.

Definition 2.5. A set $\{Q_{\boldsymbol{n}}\}$ of polynomials of index $\boldsymbol{n} \in \mathbb{N}_{s+1}$

$$\deg Q_{\boldsymbol{n}} \leqslant |\boldsymbol{n}| = \sum_{\alpha=1}^{s+1} n_\alpha$$

is called semiclassical multiple orthogonal polynomials of class S if they satisfy the $s+1$ families of orthogonality relations

$$\int_{\Gamma_\alpha} Q_{\boldsymbol{n}}(z) z^\nu w(z)\, dz = 0, \quad \nu = 0, \dots, n_{\alpha-1}, \quad \alpha = 1, \dots, s+1 \tag{2.9}$$

with respect to the same semiclassical weight $w(z)$ of class S (i.e. satisfying (2.4), (2.6)), placed on $s+1$ different curves Γ_α, $\alpha = 1, \dots, s+1$ which satisfy (2.5) and such that (w, Γ_α) are $s+1$ linearly independent semiclassical moments functionals.

Remark. It is possible to prove that $\deg Q_{\boldsymbol{n}} = |\boldsymbol{n}|$ and this definition gives the unique up to normalization sequence of polynomials.

Now semiclassical multiple orthogonal polynomial can be easily classified in the same way as classical orthogonal polynomial, i.e. just by position of singularities of the Pearson equation (2.4). For the case $S = 1$ we have

$$\begin{array}{lll}
(J-J) & \Phi(z) = z(z-a_1)(z-a_2), & w(z) = z^{\alpha_0}(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}, \\
(J-B) & \Phi(z) = z^2(z-a), & w(z) = z^{\alpha_0}(z-a)^{\alpha_1} \exp\{\gamma/z\}, \\
(B-B) & \Phi(z) = z^3, & w(z) = z^{\alpha} \exp\{\gamma_1/z^2 + \gamma_2/z\}, \\
(J-L) & \Phi(z) = z(z-a), & w(z) = z^{\alpha_0}(z-a)^{\alpha_1} \exp\{\beta z\}, \\
(B-L) & \Phi(z) = z^2, & w(z) = z^{\alpha} \exp\{\beta z + \gamma/z\}, \\
(L-H) & \Phi(z) = z, & w(z) = z^{\alpha} \exp\{\beta_1 z^2 + \beta_2 z\}, \\
(H-H) & \Phi(z) = \text{const}, & w(z) = \exp\{\beta_1 z^3 + \beta_2 z^2 + \beta_3 z\}.
\end{array} \tag{2.10}$$

Combined with arbitrary appropriate paths of integration Γ_1 and Γ_2, satisfying (2.10) which, for example, are

$(J-J)$ Γ_1: Jordan arc joining the points 0 and a_1
Γ_2: Jordan arc joining the points 0 and a_2

$(J-B)$ Γ_1: Jordan arc joining the points 0 and a, such that in the neighbourhood of zero $\Re(\gamma/z)<0$
Γ_2: Jordan curve in $\mathbb{C}$, containing the point 0, such that in the neighbourhood of zero $\Re(\gamma/z)<0$.

and so on, semiclassical functionals $(w, \{\Gamma_1, \Gamma_2\})$ of class $S=1$ generate seven sequences of multiple orthogonal polynomials which, by analogy with the classical case, we call Jacobi–Jacobi (J–J) polynomials, Jacobi–Bessel polynomials (J–B), and so on.

2.2.3. Properties of semiclassical multiple orthogonal polynomials

An important feature of the semiclassical multiple orthogonal polynomials of class S is that they inherit most of the remarkable properties of the classical orthogonal polynomials. Here we present these properties for the class $S=1$. (We remark that correspondent results can be stated for an arbitrary $S\geqslant 0$.)

The main property is the Rodrigues type formula.

Theorem 2.6 (Aptekarev, Marcellán and Rocha [14]). *Let*

$$Q_n(z) := Q_{\mathbf{n}}(z), \quad \mathbf{n}\in I_D, \qquad \deg Q_n = (s+1)\cdot n, \quad n\in\mathbb{N}_0 \tag{2.11}$$

be a diagonal semiclassical multiple orthogonal polynomial of class $S=1$. *Then*

$$Q_n(z) = \frac{1}{w(z)} D^n[\Phi^n(z)w(z)]. \tag{2.12}$$

Remark. We notice that the Rodrigues type formula (2.12) can be taken as a definition for the set of semiclassical multiple orthogonal polynomials of class S, where $w(z)$ is a solution of the Pearson equation (2.5). Indeed, historically the first examples of these polynomials have been generated by generalization of Rodrigues formula and they were investigated without any connection with property of multiple orthogonality (see [7, 31]).

Another important property of semiclassical multiple orthogonal polynomials is an explicit expression for the coefficients of $S+2$ order differential equation.

Theorem 2.7 (Aptekarev, Marcellán and Rocha [14]). *The diagonal semiclassical multiple orthogonal polynomials* (2.11) *of the class* $S=1$, *satisfy the differential equation*

$$\Phi^2(z)Q_n'''(z) + 2\Psi(z)\Phi(z)Q_n''(z) + A_1(z,n)Q_n'(z) + A_2(z,n)Q_n(z) = 0,$$

where

$$A_1(z,n) := \Psi(\Psi - \Phi') - \Phi\left(\frac{n(n+1)}{2}\Phi'' + (n-1)\Psi'\right),$$

$$A_2(z,n) := -(\Psi - \Phi')\left(\frac{n(n-1)}{2}\Phi'' + n\Psi'\right) - \Phi\left(\frac{(n)(n-1)(2n+5)}{6}\Phi''' + \frac{n(n+3)}{2}\Psi''\right).$$

Among other properties of classical orthogonal polynomials which are also valid for semiclassical multiple orthogonal polynomials we mention.

($*$) the explicit expression of the coefficients of $s+3$ recurrence relations

($**$) the explicit expression for generating function and the possibility to obtain strong asymptotics when $n \to \infty$.

A four-terms recurrence relation for the special case $w(z)=1$ of (J–J) polynomials $\boldsymbol{n} \in I_{SL}$ has been studied in [35]. It would be useful to obtain a table of explicit expressions of the coefficients of the recurrence relation for all families of semiclassical multiple orthogonal polynomials of class $S=1$.

It seems that the first result on strong (Szegö type) asymptotics for multiple orthogonal polynomials was obtained in 1979 by Kalyagin. He proved [33] the formulae of strong asymptotics for the special case of (J–J) polynomials: $w(x)$ as in (2.10) with $a_1 = -a_2 = a$, $\gamma_1 = [-a, 0]$, $\gamma_2 = [0, a]$. Strong asymptotics for the (J–B) polynomials was investigated in [14]. Sorokin studied analytic properties (including strong asymptotics) for (J–L) (case $\alpha < 0$, $\beta < 0$), (L–H) and (H–H) polynomials (see [51, 54, 56]). It looks that the cases of the (B–B) and the (B–L) polynomials have not been touched yet.

Summarizing, we would like to mention that the notion of multiple orthogonality (in comparison with the standard orthogonality) seems to be very appropriate for the generalization of classical orthogonal polynomial. The semiclassical multiple orthogonal polynomial are easily classified; they reflect the multidimensionality features of the set of semiclassical moment functionals of class S. The semiclassical multiple orthogonal polynomials inherit most of the remarkable properties of the classical orthogonal polynomials. As a consequence, the powerful tools which have been created for the treatment of classical orthogonal polynomials may be used (after adaptation) for an analysis of semiclassical multiple orthogonal polynomials. This gives the opportunity for the theory of semiclassical multiple orthogonal polynomials to develop as far as the theory of classical orthogonal polynomials. At the same time, the semiclassical multiple orthogonal polynomials show new interesting phenomena which have not occurred in the theory of classical orthogonal polynomials. They present a new set of special functions which is very important in several applications.

2.3. Spectral theory of nonsymmetric operators

Another field of application of multiple orthogonal polynomials is spectral analysis of high order difference operators, given in the standard basis

$$e_k = (\underbrace{0, 0, \ldots, 0}_{k \text{ times}}, 1, 0, \ldots), \quad k \in \mathbb{N}_0, \tag{2.13}$$

of the Hilbert space $l_2(\mathbb{N}_0)$ by a band matrix

$$A \sim (a_{ij})_{i,j=0}^{\infty}, \qquad a_{ij} = 0, \quad j > i + q, \ i > j + p \tag{2.14}$$

with nonzeros extreme diagonals

$$a_{n,n-p} \neq 0, \qquad a_{n,n+q} \neq 0.$$

Here we just touch an idea of the approach, referring for details to the recent comprehensive survey on the topic in [10].

If A is a symmetric tridiagonal matrix with real coefficients and positive extreme diagonals (i.e. Jacobi matrix)

$$\Im a_{ii} = 0, \qquad a_{i,i+1} = a_{i+1,i} > 0,$$

such that moments problem associated with A is determined, then, due to the Stone theorem, the class of the closed operators

$$Ae_n = e_{n-1}a_{n,n-1} + e_n a_{n,n} + e_{n+1}a_{n,n+1}$$

coincides with the class of Lesbegue (simple spectrum), self-adjoint operators, and by the spectral theorem of Von Neumann there exists a unique operator valued measure E_λ, such that

$$A = \int_{\mathbb{R}} \lambda \, \mathrm{d}E_\lambda.$$

The positive Borel measure

$$\mu(\lambda) := \langle E_\lambda e_0, e_0 \rangle$$

is called the spectral measure for the operator A, and the function

$$\hat{\mu}(z) := \langle R_z e_0, e_0) \rangle \qquad (*)$$

is called the resolvent (or Weyl) function; here R_z

$$R_z := (zI - A)^{-1} = \int_{\mathbb{R}} \frac{\mathrm{d}E_\lambda}{z - \lambda}$$

is the resolvent operator. For the self-adjoint operator, the resolvent function becomes the Markov type function

$$\hat{\mu}(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(\lambda)}{z - \lambda}.$$

(For details of the spectral theorem for self-adjoint operators and connection with Jacobi matrices see, for example [1, 2, 45].)

A connection between the spectral theory of self-adjoint operators and theory of orthogonal polynomials is established by Tchebyshev–Favard theorem which states that polynomial solutions $q_n(\lambda)$ of the difference equation

$$\lambda y_n = a_{n,n-1}y_{n-1} + a_{n,n}y_n + a_{n,n+1}y_{n+1} \qquad (2.15)$$

with the initial conditions

$$q_0 = 1, \qquad q_n = 0, \quad n < 0 \qquad (2.16)$$

are the orthogonal polynomials with respect to the spectral measure

$$\int_{\mathbb{R}} q_n(\lambda)\lambda^{\nu}\,\mathrm{d}\mu(\lambda)=0, \quad \nu=0,\ldots,n-1. \tag{2.17}$$

Moreover, the ratio

$$\pi_n(z)=\frac{p_n(z)}{q_n(z)}$$

of $q_n(z)$ and another linearly independent solution of (2.15) with initial condition

$$p_1=\frac{1}{a_{0,1}}, \quad p_n=0, \;\; n<0 \tag{2.18}$$

is the diagonal Pade approximant for the resolvent function (∗)

$$\hat{\mu}(z)-\pi_n(z)=\frac{c_n}{z^{2n+1}}+\cdots . \tag{2.19}$$

In general, for nonself-adjoint (nonsymmetric) operators, the notion of spectral positive measure loses sense. However, properly chosen rational approximants for the resolvent functions (∗) of operator A are still connected with the coefficients of operator (2.14). It can be illustrated with an example of an operator A given by general nonsymmetric $p+2$ diagonal matrix (2.14), with $q=1$.

The spectral problem for A leads in this case to the following difference equation of order $p+1$:

$$zy_n=a_{n,n-p}y_{n-p}+a_{n,n-p+1}y_{n-p+1}+\cdots+a_{n,n}y_n+a_{n,n+1}y_{n+1}. \tag{2.20}$$

Let $q_n(z)$, $p_n^{(j)}(z)$, $j=1,2,\ldots,p$ be the $p+1$ linearly independent solutions of (2.20) defined by the initial conditions (2.16) and

$$p_j^{(j)}=\frac{1}{a_{j-1,j}}, \qquad p_n^{(j)}=0, \quad n<j, \;\; j=1,2,\ldots,p.$$

The connection between the spectral problem and the Hermite–Padé approximants for the system of resolvent functions

$$f_j(z)=\langle R_z e_{j-1},e_0\rangle, \quad j=1,\ldots,p \tag{2.21}$$

is given by the following

Theorem 2.8 (Kaliaguine [32]). *For $n=kp+s$ the vector of rational functions*

$$\pi_n:=\left(\frac{p_n^{(1)}(z)}{q_n(z)},\ldots,\frac{p_n^{(p)}(z)}{q_n(z)}\right) \tag{2.22}$$

is the Hermite–Padé approximant of index $(k+1,\ldots,k+1,k,\ldots,k)$ for the system (2.21) *of resolvent functions* (2.21)

Thus, for nonsymmetric band operators, instead of the notion of orthogonal polynomials with respect to the positive spectral measure supported by the real line it is possible to use Hermite–Padé polynomials i.e. multiple orthogonal polynomials with respect to complex weight functions on the contours of complex plane.

As an example of application of the method of rational approximants for the resolvent functions to the spectral problem of nonsymmetric operators, we mention a criterion for the spectrum in terms of the growth of the polynomials which are numerators and denominators of the approximants. For the tridiagonal nonsymmetric operators, the criterion based on the Padé approximants (2.19) was proved in [11]. For the $p+2$ diagonal operator (2.14), $(q=1)$, the spectrum was characterised in terms of Hermite–Padé polynomials in [34]. Finally, in [17], the criterion was obtained for general band operators (2.14) by means of Hermite–Padé polynomials with matrix coefficients.

Another examples of the applications are considered in [12] where the classical Stieltjes approach has been developed and adapted for the spectral analysis of the operator (2.14) with two nonzero extreme diagonals

$$a_{i+1,i}=1, \qquad a_{i-p,i}=b_i, \qquad a_{i,j}=0, \quad i-j\neq 1 \text{ and } i-j\neq p.$$

Among other results, in [12] a procedure for integration of high order discrete KdV equation

$$\dot{b}_n=b_n\cdot\sum_{j=1}^{p}(b_{n+j}-b_{n-j}), \quad n\in\mathbb{N}, \quad b_0(t)=1,\ b_{-n}=0$$

was obtained.

An interesting open problem is to develop the scattering theory (see [15, 26]) for nonsymmetric operator based on the strong asymptotics of multiple orthogonal polynomials.

3. Asymptotics of multiple orthogonal polynomials

In this chapter we review some results and methods of obtaining asymptotics of multiple orthogonal polynomials connected with systems of Markov functions: Angelesco and Nikishin systems

$$Q_{\boldsymbol{n}}\to ?, \quad |\boldsymbol{n}|\to\infty.$$

Firstly, for the case of usual orthogonal polynomials we recall a classification of different types of asymptotics. Then we will present analogous results for multiple orthogonal polynomials.

3.1. *Orthogonal polynomials; types of asymptotics*

For the case $p=1$ and F is an interval, multiple orthogonal polynomials become the standard orthogonal polynomials

$$q_n(z)=z^n+\dots: \int_F q_n(x)x^k\,\mathrm{d}\sigma(x)=0, \quad k=0,\dots,n-1. \tag{3.1}$$

It is useful to distinguish several types of asymptotic formulas when $n\to\infty$.

3.1.1. *Weak (nth root) asymptotics*

Two following equivalent forms of asymptotics

$$|q_n(z)|^{1/n}\rightrightarrows ?, \quad z\in K\Subset\overline{\mathbb{C}}\setminus F \tag{3.2}$$

and

$$\nu_n \xrightarrow{*} ?, \quad n\to\infty, \tag{3.3}$$

where

$$\nu_n(z) := \nu[q_n] := \frac{1}{n}\sum_{j=1}^{n}\delta(z-z_{j,n}), \qquad q_n(z)=\prod_{j=1}^{n}(z-z_{j,n}),$$

and δ is the Dirac's delta function, give a rather rough description of the asymptotic behaviour of the sequence of polynomials. For the polynomials (3.1) orthogonal with respect to a measure, satisfying the condition

$$\sigma'(x)>0, \quad x\in F \text{ a.e.} \tag{3.4}$$

formulae of weak asymptotic have the forms (see for example [58])

$$|q_n(z)|^{1/n} \rightrightarrows |c\cdot\Phi(z)| \tag{3.5}$$

and

$$\nu_n \xrightarrow{*} \overset{\circ}{\lambda}_F, \quad n\to\infty, \tag{3.6}$$

where the functions, standing on the right-hand sides admit several equivalent descriptions.

(a) *Logarithmic potential form of weak asymptotics.* Let $V_\mu(z)$ be a logarithmic potential of measure μ

$$V_\mu(z)=\int \ln\frac{1}{|z-x|}\,\mathrm{d}\mu(x)$$

and let $\overset{\circ}{\lambda}_F$ be an equilibrium measure of the interval F. This measure is uniquely defined by

$$\overset{\circ}{\lambda}_F\colon V_{\overset{\circ}{\lambda}_F}(x)=\text{const}=:\gamma, \ x\in F. \tag{3.7}$$

Then on the right-hand side of (3.6) stands equilibrium measure of F (or Tchebyshev measure)

$$\mathrm{d}\lambda^\circ{}_{[-1,1]}=\frac{\mathrm{d}x}{\pi\sqrt{1-x^2}}.$$

Moreover on the right-hand side of (3.5) we have

$$|\Phi(z)|=\exp\{-V_{\overset{\circ}{\lambda}_F}(z)\}, \quad c=e^{\gamma}.$$

(b) *Riemann surface for weak asymptotics.* Let $\mathscr{R}$ be a Riemann surface with two sheets $\mathscr{R}_0$, $\mathscr{R}_1$ cut along the interval F and pasted together along the cut F in the usual crosswise way. Since genus of $\mathscr{R}$ is zero, we can consider on $\mathscr{R}$ rational function $\Psi(z)$ with divisor

$$\Psi_{(z)}\colon \begin{cases} \Psi(z)=\dfrac{1}{c_0 z}+\cdots, & z\to\infty^{(0)} \\ \Psi(z)=\dfrac{z}{c_1}+\cdots, & z\to\infty^{(1)} \end{cases} \qquad c_0\cdot c_1=1, \ c_1>0. \tag{3.8}$$

Then $\Phi(z):=\Psi(z^{(1)})$ and for $F=[-1,1]$ we have $\Phi(z)=z+\sqrt{z^2-1}$.

(c) *Boundary value problem for weak asymptotics.* The function $\Phi(z)$ can also be described as the function solving the following boundary value problem:

$$\begin{aligned}&(1)\, \Phi\in H(\mathbb{C}\setminus F), \Phi(z)\neq 0 \quad \text{for } z\in\mathbb{C}\setminus F,\\ &(2)\, \Phi(z)=z/c+\cdots, \quad z\to\infty \text{ and } c>0,\\ &(3)\, |\Phi(x)|=1, \quad x\in F.\end{aligned} \tag{3.9}$$

3.1.2. Asymptotics of $\overline{q}_n^2(x)\,\mathrm{d}\sigma(x)$ (R-asymptotics)

If we denote as $\overline{q}_n$ the orthonormal polynomial:

$$\overline{q}_n(z)=\varkappa_n q_n(z), \qquad \int_F \overline{q}_n^2(x)\,\mathrm{d}\sigma(x)=1 \tag{3.10}$$

then, as it was proved by Rakhmanov in [52], the condition (3.4) is sufficient for

$$\overline{q}_n^2(x)\,\mathrm{d}\sigma(x) \overset{*}{\longrightarrow} \mathrm{d}\mathring{\lambda}_F(x) \tag{3.11}$$

This asymptotic formula is more precise than the formula of weak asymptotics (3.5). For instance, (3.11) is equivalent to the ratio asymptotics of orthogonal polynomials. We will call the asymptotic formulae of type (3.11) as R-asymptotics.

3.1.3. Strong (Szegö type) asymptotics

Finally, the most precise description of the behaviour of the orthogonal polynomials with $n\to\infty$ is given by the formulae of strong asymptotics

$$\overline{q}_n(z)=\Phi^n(z)(f(z))+o(1)), \quad z\in K\Subset\overline{\mathbb{C}}\setminus F, \tag{3.12}$$
$$\|\overline{q}_n(x)-(\Phi^n(x)f(x)+\overline{\Phi^n(x)f(x)})\|_{L_2,\sigma(F)}=o(1),$$

where f is called Szegö function and is uniquely defined by the following boundary value problem:

$$f:\quad \begin{aligned}&(1)\ f\in H(\overline{\mathbb{C}})\setminus F), \quad f(z)\neq 0 \quad \text{for } z\in\mathbb{C}\setminus F,\\ &(2)\ |f(x)|^2\sigma'(x)=\mathring{\lambda}_F'(x), \quad x\in F \text{ a.e.}\end{aligned} \tag{3.13}$$

The formulae of the strong asymptotics are valid when the condition

$$\int_F \ln\sigma'(x)\,\mathrm{d}\mathring{\lambda}_F(x)>\infty$$

is fulfilled (see, for example, [59]).

3.2. Weak asymptotics of multiple orthogonal polynomials

As we already mentioned in the Introduction that the asymptotic formulae of multiple orthogonal polynomials for Angelesco and Nikishin systems follow from the asymptotics of correspondent polynomials orthogonal with respect to the varying weights (see Section 1.4)

$$\int_{F_\alpha} q_{\alpha,\boldsymbol{n}}(x)x^\nu w_\alpha(\boldsymbol{q}_{\boldsymbol{n}};x)\,\mathrm{d}\sigma_\alpha(x)=0, \quad \nu=0,\ldots,n-1, \ \ \alpha=1,\ldots,p, \tag{3.14}$$

here we consider diagonal case $\mathbf{n}\in I_{\mathrm{D}}$ (see (1.6)), $q_{\alpha,\mathbf{n}}(x)=x^n+\cdots$. We remind, that for Angelesco system, the varying weight is defined by (1.19) and the monic multiple orthogonal polynomial is

$$\text{(A):}\quad Q_{\mathbf{n}}(x)=\prod_{\alpha=1}^{p} q_{\alpha,\mathbf{n}}(x). \tag{3.15}$$

For Nikishin system, the varying weight is defined by (1.22)–(1.23) and the multiple orthogonal polynomial is

$$\text{(N):}\quad Q_{\mathbf{n}}(x)=q_{1,\mathbf{n}}(x). \tag{3.16}$$

Orthogonal polynomials with respect to a varying weight, corresponding to the Angelesco and Nikishin systems, possess the same types of asymptotic formulae as the standard orthogonal polynomials. For the weak asymptotic we have

$$|q_{\alpha,\mathbf{n}}(z)|^{1/n}\rightrightarrows|c_\alpha\Phi_\alpha(z)| \tag{3.17}$$

and

$$\nu\left[q_{\alpha,\mathbf{n}}\right]\xrightarrow{*}\lambda_\alpha,\quad \alpha=1,\ldots,p, \tag{3.18}$$

however the function $\Phi_\alpha(z)$ and the measure λ_α have a different meaning than in (3.5), (3.6).

3.2.1. Vector potential approach

The most powerful tool for the study of the weak asymptotics of the multiple orthogonal polynomials was introduced by Gonchar and Rakhmanov [27, 28]. This approach is based on the notion of equilibrium measure of vector potential problem. We present the necessary definitions. Let $F=\{F_\alpha\}_{\alpha=1}^{p}$ be a system of the intervals of the real axis $\mathbb{R}$, $A=(a_{\alpha,\beta})_{\alpha,\beta=1}^{p}$ be a real, symmetric, positive definite matrix. Let $\boldsymbol{\sigma}=\{\sigma_\alpha\},\alpha=1,\ldots,p$ be a vector of positive, probabilistic, Borel measures, such that $\operatorname{supp}\sigma_\alpha\subseteq F_\alpha,\ \alpha=1,\ldots,p$.

Definition 3.1. A vector

$$W_{\boldsymbol{\sigma}}=\{W_{\boldsymbol{\sigma}}^{\alpha}(x)\},\qquad W_{\boldsymbol{\sigma}}^{\alpha}(z)=\sum_{\beta=1}^{p}a_{\alpha,\beta}V_{\sigma_\beta}(z)$$

is called a vector potential of the vector measure $\boldsymbol{\sigma}$ with the matrix of interaction A.

Theorem 3.2 (Gonchar and Rakhmanov [27, 28]). *If the system of intervals F and the matrix of interaction A satisfy*

$$F_\alpha\cap F_\beta\neq\emptyset\ \Rightarrow\ a_{\alpha,\beta}=0,\quad \alpha,\beta=1,\ldots,p,$$

then there exists an unique vector measure $\boldsymbol{\lambda}=\boldsymbol{\lambda}(F,A)=\{\lambda_\alpha\}_{\alpha=1}^{p}$ such that for the vector potential of this measure the following equilibrium conditions hold

$$W_{\boldsymbol{\lambda}}^{\alpha}(x)\begin{cases}=\gamma_\alpha, & x\in\operatorname{supp}\lambda_\alpha=:F_\alpha^*\\ \geqslant\gamma_\alpha, & x\in F_\alpha\setminus F_\alpha^*\end{cases}\qquad \alpha=1,\ldots,p. \tag{3.19}$$

Definition 3.3. The vector measure $\lambda(F,A)$ is called the equilibrium measure for the vector potential problem (F,A).

Now we can state the results about weak asymptotics. For the Angelesco system we have

Theorem 3.4 (Gonchar and Rakhmanov [27]). *If σ_α in (3.11) satisfy condition (3.4), then for polynomials $q_{\alpha,n}$ (3.14) corresponding to the Angelesco system (i.e. for w_α (1.19) holds), weak asymptotic formulae (3.18) are valid, where λ_α are the components of the equilibrium measure of the vector potential problem (F,A) with matrix of interaction*

$$A=\begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix}. \tag{3.20}$$

Analogous theorem for Nikishin system (with several degree of generality) has been proved in [20, 25, 29, 44], the only difference that for Nikishin system, the matrix of interaction has a form

$$A=\begin{pmatrix} 2 & -1 & 0 & & & \cdots & 0 \\ -1 & 2 & -1 & 0 & & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & & \cdots & -1 & 2 & & -1 \\ 0 & & \cdots & 0 & -1 & & 2 \end{pmatrix}. \tag{3.21}$$

It is important for several applications to know information about structure of the supports of the components of the equilibrium measure. For the Angelesco systems the supports of vector equilibrium measure (3.19)–(3.20) are intervals $F_\alpha^* \subseteq F_\alpha$ not necessarily coinciding with input system of interval F_α, $\alpha=1,\dots,p$ [27]. For Nikishin systems (see, for example, [29]) intervals $F_\alpha^*=F_\alpha$, $\alpha=1,\dots,p$, holds. For the multiple orthogonal polynomials corresponding to the mixed Angelesco–Nikishin systems defined by a graph-tree, the problem of weak asymptotics has been solved in [29]. It still remains an interesting open problem to describe the structure of the supports of the equilibrium measure for this case.

3.2.2. Riemann surface and matrices of interaction

Like for the case of standard orthogonal polynomials, functions Φ_α, $\alpha=1,\dots,p$, defining the weak asymptotics in (3.17) can be described as branches of a rational function on some Riemann surface. The approach to the asymptotics of multiple orthogonal polynomials based on analysis of the Riemann surface has been introduced by Nuttall [46, 47] (see also [21]).

Let us define Riemann surfaces governing the asymptotics for the cases of the Angelesco and Nikishin systems. For both cases they are Riemann surfaces with $p+1$ sheets and square root's branch points at the end points of intervals $F_\alpha^*=[a_\alpha^*,b_\alpha^*]$, which are the supports of the vector equilibrium measure (3.19) (we remind that for Nikishin systems $F_\alpha^*=F_\alpha, \alpha=1,\dots,p$). The difference appears in the matrices of monodromy corresponding to the branch points. The situation is similar to the difference between the matrices of interaction of vector potential problem for Angelesco and Nikishin systems. For the Angelesco system, the matrices of monodromy are

$$\text{(A)}: \quad \mathcal{M}_{a_\alpha^*} = \mathcal{M}_{b_\alpha^*} := E_{0,\alpha}, \quad \alpha = 1, \ldots, p, \tag{3.22}$$

where $E_{i,j}$ is the matrix swapping the ith and the jth coordinates of a vector, and for the Nikishin systems, the matrices of monodromy are

$$\text{(N)}: \quad \mathcal{M}_{a_\alpha} = \mathcal{M}_{a_\alpha} := E_{\alpha-1,\alpha}, \quad \alpha = 1, \ldots, p. \tag{3.23}$$

Thus for both cases we have defined $p+1$ sheets Riemann surfaces

$$\mathcal{R} = \mathcal{R}(F^*, \mathcal{M}). \tag{3.24}$$

Similar to (3.8) on $\mathcal{R}$ a rational function $\Psi(z)$ can be defined by divisor

$$\Psi(z): \begin{cases} \Psi(z) = 1/c_0 z + \cdots & z \to \infty^{(0)} \\ \Psi(z) = z/c_1 + \cdots & z \to \infty^{(1)} \\ \cdots & \cdots \\ \Phi(z) = z/c_p + \cdots & z \to \infty^{(p)} \end{cases} \quad c_1 > 0, \ \prod_{\alpha=1}^{p} c_\alpha = 1. \tag{3.25}$$

Now we can express the main term of the asymptotics (see (3.17)) by means of branches of the algebraic function Ψ. For the Angelesco systems (see [13]) from Theorem 3.4 it follows that

$$\text{(A)}: \quad \Phi_\alpha(z) = \Psi(z^{(\alpha)}), \quad \alpha = 1, 2, \ldots, p. \tag{3.26}$$

For the Nikishin systems (see [9]) the relation

$$\text{(N)}: \quad \Phi_\alpha(z) = \prod_{j=\alpha}^{p} \Psi(z^{(j)}), \quad \alpha = 1, 2, \ldots, p \tag{3.27}$$

is valid.

3.2.3. *Boundary value problems for the main term*

From (3.26) and (3.27) it follows that the main term of asymptotics solves the following boundary value problem, generalizing (3.9). For an Angelesco system it is

$$\begin{aligned} &(1)\ \Phi_\alpha \in H(\mathbb{C} \setminus F_\alpha^*) \quad \text{and} \quad \Phi_\alpha(z) \neq 0 \ \text{ for } z \in \mathbb{C} \setminus F_\alpha^*, \\ &(2)\ \Phi_\alpha(z) = z/c_\alpha + \cdots \ \ c_\alpha > 0, \\ &(3)\ |\Phi_\alpha(x)|^2 \prod_{\beta \neq \alpha} |\Phi_\beta(x)| = 1 \ \ x \in F_\alpha^*, \ \ \alpha = 1, \ldots, p. \end{aligned} \tag{3.28}$$

The corresponding boundary value problem for a Nikishin system is

$$\begin{aligned}
&(1)\ \Phi_\alpha \in H(\mathbb{C}\backslash F_\alpha) \quad \text{and} \quad \Phi_\alpha(z) \neq 0 \ \text{ for } z \in \mathbb{C}\backslash F_\alpha,\\
&(2)\ \Phi_\alpha(z) = z^{p+1-\alpha}/c_\alpha + \cdots \ c_\alpha > 0,\ \ \alpha = 1,2,\ldots,p,\\
&(3)\ |\Phi_\alpha(x)|^2 \frac{1}{|\Phi_{\alpha-1}(x)\Phi_{\alpha+1}(x)|} = 1, \quad x \in F_\alpha,\ \ (\Phi_0 \equiv \Phi_{p+1} \equiv 1).
\end{aligned} \tag{3.29}$$

3.3. R-asymptotics for polynomials orthogonal with varying weight

If we consider normalized polynomials $\tilde{q}_{\alpha,n}, \alpha = 1,\ldots,p$, orthogonal with respect to the varying weight $\tilde{w}_{\alpha,n}(x) := \beta_{\alpha,n} w_\alpha(\boldsymbol{q}_n, x)$:

$$\tilde{q}_{\alpha,n} := \varkappa_{\alpha,n} q_{\alpha,n},\colon \int_{F_\alpha} \tilde{q}^2_{\alpha,n}(x) \tilde{w}_{\alpha,n}(x)\, \mathrm{d}\sigma_\alpha(x) = 1, \tag{3.30}$$

then we can pose a question about R-asymptotics (see Section 3.1.2)

$$\tilde{q}^2_{\alpha,n}(x) \tilde{w}_{\alpha,n}(x)\, \mathrm{d}\sigma_\alpha(x) \xrightarrow{*} \mathrm{d}\mathring{\lambda}_{F_\alpha^*}(x), \quad \alpha = 1,\ldots,p. \tag{3.31}$$

We emphasize that on the right-hand side of (3.31) we have the usual equilibrium measure (3.7) of the interval F^*_α (not the component of the vector equilibrium measure (3.19)). Thus, similar to the case of standard orthogonal polynomials, it is interesting to prove R-asymptotics of the polynomials orthogonal with respect to the varying weights for the measures $\mathrm{d}\sigma_\alpha(x)$ satisfying to the conditions (3.4).

As it follows from [37], for the polynomials orthogonal with respect to the varying weight corresponding to the Nikishin systems (see (1.21)–(1.23)) condition (3.4) is sufficient for validness of (3.31). Moreover, in this case, for the integral component (1.23) of the varying weight (1.22)

$$h_{\alpha,\boldsymbol{n}}(z) = \int_{F_{\alpha-1}} \frac{\tilde{q}^2_{\alpha-1,n}}{z-x} \frac{h_{\alpha-1,\boldsymbol{n}(x)}\, \mathrm{d}x}{\tilde{q}_{\alpha-2,n}(x) \tilde{q}_{\alpha,n}(x)}, \quad h_{1,\boldsymbol{n}} \equiv 1, \ \ \alpha = 2,\ldots,p,$$

we have uniform asymptotics

$$h_{\alpha,\boldsymbol{n}}(z) \rightrightarrows \int_{F_{\alpha-1}} \frac{\mathrm{d}\mathring{\lambda}_{F_{\alpha-1}}(x)}{z-x} =: \hat{\lambda}_{F_{\alpha-1}}(z), \quad \alpha = 2,\ldots,p. \tag{3.32}$$

To prove (3.31) under condition (3.4) for the Angelesco system is still an *open problem.*

3.4. Strong asymptotics for mulitple orthogonal polynomials

Orthogonality relations (1.17), (1.19) for the Angelesco system are equivalent to the system of extremal problems

$$\int_{F_\alpha} q^2_\alpha(x) \prod_{\beta \neq \alpha} q_\beta(x)\, \mathrm{d}\mu_\alpha(x) = \min_{P(x) = x^n + \cdots} \int_{F_\alpha} P^2(x) \prod_{\beta \neq \alpha} q_\beta(x)\, \mathrm{d}\mu_\alpha(x). \tag{3.33}$$

We have the same for the Nikishin systems

$$\int_{F_\alpha} q_\alpha^2(x)\frac{h_\alpha(x)}{(q_{\alpha-1}q_{\alpha+1})(x)}\,\mathrm{d}\sigma_\alpha(x)=\min_{P(x)=x^{n(p+1-\alpha)}+\cdots}\int_{F_\alpha} P^2(x)\frac{h_\alpha(x)}{(q_{\alpha-1}q_{\alpha+1})(x)}\,\mathrm{d}\sigma_\alpha(x). \tag{3.34}$$

Taking into account the boundary properties of the main term of asymptotics (3.28), (3.29) from (3.33), (3.34) we can derive a heuristic conclusion, that the next term of the asymptotics

$$\frac{q_{\alpha,n}}{(c_\alpha\Phi_\alpha)^n}\simeq F_\alpha+\mathrm{o}(1)$$

should satisfy

$$F_\alpha(z)=\frac{f_\alpha(z)}{f_\alpha(\infty)}, \tag{3.35}$$

where $f_\alpha(z)$ is a solution of the following system of boundary value problems for the Angelesco case:

$$f_\alpha(z):\quad \begin{array}{l}(1)\ f_\alpha\in H(\mathbb{C}\backslash F_\alpha),\quad f_\alpha(z)\neq 0,\ f_\alpha(\infty)>0,\\ (2)\ |f_\alpha(x)|^2\prod\limits_{\beta\neq\alpha}|f_\beta(x)|\mu_\alpha'(x)=\mathring{\lambda}'_{F_\alpha^*}(x),\quad x\in F_\alpha^*,\ \ \alpha=1,2,\ldots,p\end{array} \tag{3.36}$$

and analogous system for the Nikishin case (here we also take into account (3.32))

$$f_\alpha(z):\quad \begin{array}{l}(1)\ f_\alpha\in H(\mathbb{C}\backslash F_\alpha),\quad f_\alpha(z)\neq 0,\quad f_\alpha(\infty)>0,\ \alpha=1,\ldots,p,\\ (2)\ |f_\alpha(x)|^2\dfrac{\hat{\lambda}_{F_{\alpha-1}}(x)\sigma_\alpha'(x)}{|(f_{\alpha+1}f_{\alpha-1})(x)|}=\mathring{\lambda}'_{F_\alpha}(x),\quad x\in F_\alpha,\ f_0\equiv f_{p+1}\equiv\hat{\lambda}_{F_0}\equiv 1.\end{array} \tag{3.37}$$

In [8, 9] it has been proved that under Szegö condition

$$\begin{array}{ll}\text{(A)} & \displaystyle\int_{F_\alpha^*}\ln\mu_\alpha'(x)\,\mathring{\lambda}'_{F_\alpha^*}(x)\,\mathrm{d}x>-\infty,\\ \text{(N)} & \displaystyle\int_{F_\alpha}\ln\sigma_\alpha'(x)\,\mathring{\lambda}'_{F_\alpha}(x)\,\mathrm{d}x>-\infty\end{array} \tag{3.38}$$

boundary value problems (3.36) and (3.37) have a unique solution.

Now we are prepared to state theorems about strong asymptotics proven in [8] and [9].

Theorem 3.5. *If* $\{\mu_\alpha\}_{\alpha=1}^p$, $\{\sigma_\alpha\}_{\alpha=1}^p$ *generating the Angelesco and the Nikishin systems* (1.12), (1.13)–(1.14) *satisfy to the Szegö condition* (3.38), *then for the polynomials* $q_\alpha(z)$ *orthonormal with respect to the varying weights corresponding to the Angelesco* (1.19) *or to the Nikishin* (1.22)–(1.23) *systems, the following formulae of the strong asymptotics are valid*:

$$\frac{q_\alpha(z)}{(c_\alpha\Phi_\alpha(z))^n}\rightrightarrows\frac{f_\alpha(z)}{f_\alpha(\infty)},\quad z\in K\Subset\overline{\mathbb{C}}\backslash F_\alpha^*, \tag{3.39}$$

$$\left\|\frac{q_\alpha(x)}{|c_\alpha\Phi_\alpha(x)|^n}-\left\{\left(\frac{\Phi_\alpha(x)}{|\Phi_\alpha(x)|}\right)^n\frac{f_\alpha(x)}{f_\alpha(\infty)}+\overline{\left(\frac{\Phi_\alpha(x)}{|\Phi_\alpha(x)|}\right)^n\frac{f_\alpha(x)}{f_\alpha(\infty)}}\right\}\right\|_{L^2[F_\alpha^*,\sigma]}\to 0 \tag{3.40}$$

and

$$\frac{q_\alpha}{\tilde{q}_\alpha} \simeq \frac{c_\alpha^n}{f_\alpha(\infty)}(1+o(1)), \quad n\to\infty, \ \alpha=1,2,\ldots,p, \tag{3.41}$$

where Φ_α, c_α *and* f_α *for the Angelesco systems are defined in* (3.28) *and* (3.36) *and for the Nikishin systems in* (3.29) *and* (3.37).

As corollary of the theorem we have a strong asymptotic formulae for the multiple orthogonal polynomials, corresponding to the Angelesco and to the Nikishin systems

$$\text{(A)} \quad Q_n(z) \simeq \left(\prod_{\alpha=1}^{p} c_\alpha \Phi_\alpha(z)\right)^n \left(\prod_{\alpha=1}^{p} \frac{f_\alpha(z)}{f_\alpha(\infty)} + o(1)\right),$$

$$\text{(N)} \quad Q_n(z) \simeq (c_\alpha \Phi_1(z))^n \left(\frac{f_1(z)}{f_1(\infty)} + o(1)\right).$$

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